A Graph Coloring Problem with Applications to Data Compression

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Abstract

We study properties of graph colorings that minimize the quantity of color information with respect to a given probability distribution P on the vertices of the graph. The minimum entropy of any coloring of such a *probabilistic graph* (G, P) is the *chromatic entropy* $H_\chi(G, P)$. Applications of the chromatic entropy are found in data compression with side information at the receiver and digital image partition encoding. We show that minimum entropy colorings are hard to compute even if G is planar, a minimum cardinality coloring is given and P is uniform, but that there exists a polynomial algorithm for finding a minimum entropy coloring of complements of triangle-free graphs. We also consider the minimum number of colors $\chi_H(G, P)$ in a minimum entropy coloring, and show that it is upper-bounded by $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree, and that it is upper-bounded by $\Delta(G)$ if P is uniform. Finally, we show that $\chi_H(G, P)$ can be arbitrarily larger than the chromatic number $\chi(G)$ of the graph, even for restricted families of graphs with uniform P.

1 Introduction

Graph coloring problems are classical optimization problems to which a huge amount of literature is dedicated [Die97, JT95].

In this paper, we study hardness and properties of a weighted graph coloring problem with an information-theoretic objective function. This problem has applications in the context of data compression. We first give some useful definitions, then describe the main motivations, and summarize the contributions. We also refer to several related graph coloring problems that have been studied previously.

1.1 Definitions

We use the standard notations $V(G)$ and $E(G)$, that denote respectively the set of vertices and the set of edges of a simple undirected graph G.

Definition 1. *A* coloring *of a graph G is a map $\phi : V(G) \mapsto \mathbb{N}$ of colors to vertices of G such that any two adjacent vertices have different colors.*

The kind of weighted graph used in this paper were first described as "probabilistic" in [AO96, KO98].

Definition 2. *A* probabilistic graph *consists of a graph and a random variable distributed over its vertices. It is denoted by (G, P), where G is the graph and P the probability distribution of the random variable.*

The chromatic entropy of a probabilistic graph was first defined by Alon and Orlitsky [AO96]. It gives the minimum quantity of color information contained in any coloring of a probabilistic graph. We recall the definition of entropy.

Definition 3. *The* entropy *of a random variable $X \in \mathcal{X}$ is*

$$H(X) = \sum_{x \in \mathcal{X}} f(\Pr[X = x]),$$

where

$$f(p) = -p \log p.$$

The entropy, defined by Shannon [Sha48], quantifies the information contained in a random variable. It is shown to be equal to minimum number of bits (if we use base-two logarithms) required to transmit a random variable over a noiseless communication channel. The base of the logarithms has no importance in general.

Definition 4. *Let $P(x)$ be the probability distribution of a random variable $X \in V(G)$. The* chromatic entropy *$H_\chi(G, P)$ of the probabilistic graph (G, P) is the minimum entropy of any of its coloring:*

$$H_\chi(G, P) = \min_{\phi \text{ coloring of } G} H(\phi(X)),$$

where $\phi(X)$ is a random variable defined on $\mathbb{N}$ and such that

$$\Pr[\phi(X) = c] = \sum_{\substack{x \in V(G) \\ \phi(x) = c}} P(x).$$

We will use the notation $P(S) = \sum_{x \in S} P(x)$ for a set of vertices S. Hence $\Pr[\phi(X) = c] = P(\phi^{-1}(c))$. We can now define our optimization problem.

Definition 5. *An instance of* MinEntCol *is defined by a probabilistic graph (G, P). A feasible solution of* MinEntCol *is a coloring of the vertices in $V(G)$. An optimal solution of* MinEntCol *satisfies $H(\phi(X)) = H_\chi(G, P)$, with $V(G) \ni X \sim P(x)$.*

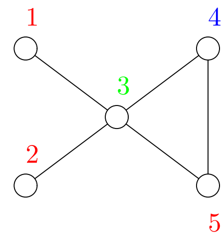
We are also interested in the number of colors defined by the following quantity.

Definition 6. *$\chi_H(G, P)$ is the minimum number of colors needed in any optimal solution of* MinEntCol.

Note that if we restrict ourselves to uniform distributions, H_χ and χ_H can be seen as two novel graph invariants.

$Y \backslash X$	1	2	3	4	5
1	1/7		1/7		
2		1/7	1/7		
3			1/7	1/7	1/7

(a) joint probability distribution



(b) corresponding characteristic graph

Figure 1: An example of joint probability distribution of two random variable X and Y, together with the corresponding characteristic graph. Coloring vertices 1, 2 and 5 in the same color and the vertices 3 and 4 in two more distinct colors yields an entropy of $\frac{6}{7}\log\frac{7}{3} + \frac{1}{7}\log 7 \simeq 1.45$ bits.

1.2 Applications

Applications of these definitions can be found in the situation in which a transmitter wishes to send a discrete random variable $X \in \mathcal{X}$ to a receiver who knows some other discrete random variable $Y \in \mathcal{Y}$ that is not independent of X. The above setting is referred to as "coding with side information at the receiver". It has been studied in the context of zero-error information theory [KO98], where nonzero error probabilities are not tolerated. Designing zero-error variable-length codes for this scheme is strongly related to the problem of finding a minimum entropy coloring of the characteristic graph G, where $V(G) = \mathcal{X}$ and two vertices are adjacent if the two symbols are undistinguishable by the receiver using the value of Y. We say that two elements x_1 and x_2 of $\mathcal{X}$ are undistinguishable if there exists a $y \in \mathcal{Y}$ such that $\Pr[X = x_1 \wedge Y = y] > 0$ and $\Pr[X = x_2 \wedge Y = y] > 0$. An illustration of this construction is given in Fig. 1. It is worth noticing that any graph G is the characteristic graph of some distribution. The minimum size of $\mathcal{Y}$ required is the minimum size of a clique cover of G.

Two families of variables-length codes for this source coding problem were defined by Alon and Orlitsky [AO96]:

restricted input (RI) codes, which are mappings of the form $\psi : V(G) \mapsto \{0,1\}^*$ such that $\psi(x)$ is not a prefix of $\psi(x')$ whenever $\{x, x'\} \in E(G)$,

unrestricted input (UI) codes, which are mappings of the form $\psi : V(G) \mapsto \{0,1\}^*$ such that $\psi(x)$ is not a proper prefix of $\psi(x')$ for all distinct pairs of vertices $\{x, x'\}$ and $\psi(x) \neq \psi(x')$ whenever $\{x, x'\} \in E(G)$.

They showed that the lowest asymptotically achievable rate for transmitting multiple instances of X using a RI code is $\lim_{n\to\infty} H_\chi(G^n, P^{(n)})/n$, where G^n is the nth AND-power of G and $P^{(n)}$ is the corresponding vector distribution. Koulgi et al. [KTRR03] gave another characterization of this rate, and NP-hardness results for the computation of optimal RI and UI codes. Zhao and Effros [ZE03] proposed a NP-hardness proof for the computation of $H_\chi(G, P)$. UI codes can be seen as a composition of a coloring of the graph G and a prefix-free encoding of this coloring, which is why the computation of H_χ is important in practice. The coloring approach is employed in both [KTRR03] and [ZE03] to design codes that are close to optimality.

Another practical application of the chromatic entropy is the compression of digital image partitions created by segmentation algorithms. Let us assume that a bitmap image has been partitioned in connected regions and that we only wish to distinguish, for each pixel, the region to which it belongs. We can assign a codeword to each pixel such that the original regions are connected regions of pixels with the same codeword. Clearly, we have an upper bound of $\log 4 = 2$ bits per pixel, since any planar map can be colored with four colors. We can however reduce the quantity of information using the minimum entropy coloring of the adjacency graph of the regions, each region being weighted by the proportion of pixels it contains. This application is described by Accame et al. [ANG00] and Agarwal and Belongie [AB02] and raises an interest in minimum entropy colorings of planar graphs.

1.3 Contributions

In section 2 we give two useful lemmas related to the entropy function and prove some elementary properties of minimum entropy colorings.

In section 3, we prove that MinEntCol is NP-hard, even if G is planar, P is uniform over $V(G)$ and a minimum cardinality coloring of G is given.

In section 4, we give a polynomial algorithm for MinEntCol when $\alpha(G) \leq 2$, hence in complements of triangle-free graphs. This includes complements of bipartite graphs and complements of cycles.

In section 5 we prove the following results concerning the number $\chi_H(G, P)$:

- $\chi_H(G, P) \leq \Delta(G) + 1$,
- a variant of Brooks' theorem for minimum entropy colorings: if G is not a complete graph nor an odd cycle and P is uniform over $V(G)$, we can always find a minimum entropy coloring of (G, P) with at most $\Delta(G)$ colors;
- there exist graphs with small chromatic number whose minimum entropy colorings require arbitrarily many colors. More precisely, $\chi_H(G, P)$ is not bounded by any function of $\chi(G)$, even if G is a planar graph or a tree and P is uniform.

The conclusion summarizes the contribution and gives some insights on how some of the results can be generalized to other graph coloring problems.

1.4 Related coloring problems

There exist a number of coloring problems using an alternative cost measure that share some similarities to the problem studied here.

In [GZ97], Guan and Zhu propose an analysis of a coloring problem on weighted graphs, in which the objective function to minimize is the sum of the maximum vertex weight in each color class. They study in particular the number of colors required, and provide some results on graphs with bounded tree-width. A similar problem is studied in [DdWMP02], with several hardness theorems.

In [BNBH$^+$98], Bar-Noy et al. study the problem of finding a vertex coloring $\phi : V(G) \to \mathbb{N}$ of a graph G such that $\sum_{v \in V(G)} \phi(v)$ is minimized. This problem is referred to as the minimum color sum problem.

Recently, Murat and Paschos [MP03] studied a probabilistic weighted graph coloring problem in which the number of colors is averaged over all possible vertex subsets. This problem is a generalization of the minimum cardinality coloring problem. They provide bounds on the objective function, several hardness results and a polynomial special case.

2 Elementary Properties

The following two lemmas are simple properties that are used in our proofs. Lemma 2 is a crucial property of the objective function that depends only on the concavity of $f : x \mapsto -x \log x$.

Lemma 1. *Let $X \sim P(x)$ and $P_{\max} = \max_x P(x)$. Then we have $H(X) \geq -\log P_{\max}$, with equality if and only if $P(x) = P_{\max}$ for all x.*

Proof.

$$\begin{aligned} H(X) &= -\sum_x P(x) \log P(x) && (1)\\ &\leq -\sum_x P(x) \log P_{\max} && (2)\\ &= -\log P_{\max} \sum_x P(x) && (3)\\ &= -\log P_{\max}. && (4) \end{aligned}$$

□

Lemma 2. *Let P be a probability distribution over $\mathbb{N}^+$ such that $P(1) \geq P(2) > 0$. Let P' be a probability distribution over $\mathbb{N}^+$ with $P'(1) = P(1) + \alpha$, $P'(2) = P(2) - \alpha$ for $0 < \alpha \leq P(2)$ and $P'(x) = P(x)$ for $x > 2$. Then if $X \sim P(x)$ and $Y \sim P'(y)$ we have $H(X) > H(Y)$.*

Proof. $f : x \mapsto -x \log x$ is a strictly concave function, hence $f(x+\alpha)-f(x)$ must decrease strictly with x. Since $P(1) \geq P(2)$ we have $f(P(1)+\alpha) - f(P(1)) < f(P(2)) - f(P(2)-\alpha)$ and

$$H(X) - H(Y) = f(P(2)) - f(P(2)-\alpha) - f(P(1)+\alpha) + f(P(1)) > 0.$$

□

We now provide a property of minimum entropy colorings that establishes the link between the uniform and nonuniform cases.

Property 1. *For any probabilistic graph (G, P) with P rational, we can construct a graph G' such that $H_\chi(G, P) = H_\chi(G', U)$, where U is the uniform distribution on $V(G')$, and such that the same number of colors is required to color (G, P) and (G', U) with minimum entropy.*

Proof. Each vertex x of $V(G)$ has a probability a_x/b, where a_x and b are integer values, and b is the least common denominator of the values $P(x) \in \mathbb{Q}$. The set $V(G')$ is obtained by replacing each vertex x by a_x copies of itself, and has size $|V(G')| = \sum_{x \in V(G)} a_x$. The duplicated vertices in G' are adjacent to the same vertices as in G, and also to all the copies of these vertices. Hence adjacent vertices in G correspond to bipartite cliques in G'.

To translate a minimum entropy coloring of (G, P) in a minimum entropy coloring of (G', U), we assign the color of $x \in V(G)$ to all a_x copies of x in $V(G')$. By definition of G', the entropy of this coloring for (G', U) is the same as the entropy of the coloring of (G, P). Also, the entropy is minimized among all colorings that assign the same colors to the duplicated vertices. It remains to show that this coloring of (G', U) is globally optimal, i.e., if two duplicated vertices have different colors, then the coloring cannot have minimum entropy.

Let us assume that two duplicated vertices $x, y \in V(G')$ have different colors in a minimum entropy coloring of (G', U). Let us further assume that c is the color of x and that there are no less vertices colored with c than with the color of y. Then coloring y with c is allowed, because x and y have the same set of neighbors, and, from Lemma 2, must result in a net decrease in entropy. Hence the coloring has not minimum entropy. □

From Lemma 2, we can see that large color classes should be favored. We can ask ourselves the question of whether the maximum independent set always form a color class in any minimum entropy coloring.

Property 2. *An optimal solution of* MinEntCol *does not always contain a maximum weight independent set as color class.*

The bipartite planar graph shown on Fig. 2 is a counterexample that invalidates this conjecture, formulated in [AB02].

It would be also interesting to know whether we can restrict our investigations to connected graphs. The following property gives a negative answer.

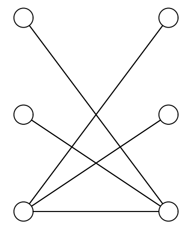


Figure 2: A counterexample to the conjecture in [AB02]: there is a unique maximum independent set of size four, and coloring all its vertices in the same color, with P uniform, leads to an entropy of 1.2516 bit. But an entropy of 1 bit is achievable with a 2-coloring.

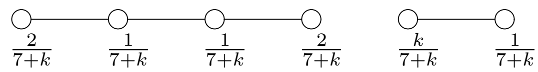


Figure 3: Example for property 3.

Property 3. *Let ϕ be a minimum entropy coloring of a probabilistic graph (G, P). If F is a disjoint component of G, then the coloring ϕ restricted to F is not always a minimum entropy coloring of (F, P'), where P' is the conditional distribution over the vertices of F defined from P.*

The graph shown on Fig. 3 is a counterexample. The chromatic entropy for the lefthand component of the graph is achieved with two colors: from property 1, this is actually the same graph as in Fig. 2 with respect to our problem. Combined with a two-coloring of the righthand component, the two-coloring of the first component yields a probability distribution of the form $\left(\frac{3+k}{7+k}, \frac{4}{7+k}\right)$. On the other hand, if we use three colors for the lefthand component, we have a probability distribution of the form $\left(\frac{4+k}{7+k}, \frac{2}{7+k}, \frac{1}{7+k}\right)$. The entropy of the latter is always smaller for sufficiently large values of k.

3 Hardness

A weaker hardness theorem for the computation of $H_\chi(G, P)$ was already given recently in [ZE03] that did not apply to the uniform case; our proof does, and is much simpler.

Theorem 1. MinEntCol *is NP-hard, even if P is the uniform distribution, G is planar, and a minimum cardinality coloring of G is given.*

Proof. Let F be a planar graph with $n \geq 1$ vertices, and let G be the graph obtained from F by attaching a triangle to each vertex and adding a vertex-

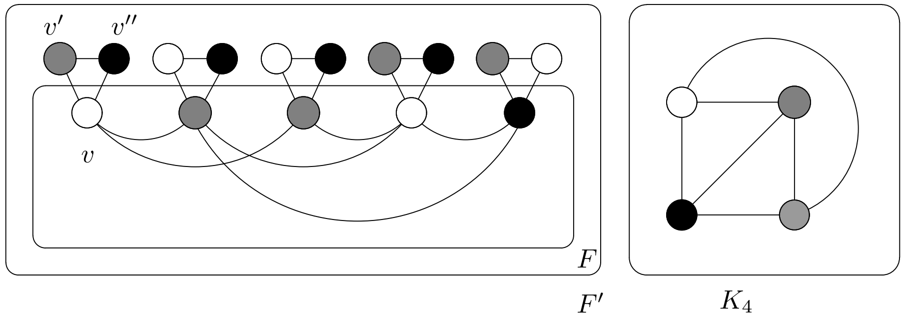


Figure 4: Illustration of the construction in Theorem 1.

disjoint 4-clique to the resulting graph. In other words, let G be the graph defined as $G = F' \cup K_4$, where $V(F') = \{v, v', v'' : v \in V(F)\}$, $E(F') = \{\{v, v'\}, \{v', v''\}, \{v'', v\} : v \in V(F)\}$, and K_4 is a 4-clique with $V(F') \cap V(K_4) = \emptyset$. The construction is illustrated on Fig. 4.

Because any maximal independent set of G contains at most n vertices from F' and one vertex from K_4, the size of a maximum independent set in G is $\alpha(G) = n + 1$.

Since F is planar, F' remains so, and so does G. Finding a 4-coloring of G can be done in quadratic time from the 4-color theorem. This 4-coloring has minimum cardinality, since K_4 requires four colors. Hence $\chi(G) = 4$.

Consider the problem of computing $H_\chi(G, U)$, where U is the uniform distribution on $V(G)$. There exists a coloring that induces the following color probabilities:

$$\left(\frac{n+1}{3n+4}, \frac{n+1}{3n+4}, \frac{n+1}{3n+4}, \frac{1}{3n+4}\right),$$

iff F is 3-colorable. Since $n + 1$ is the size of a maximum independent set in G, this distribution can be shown, from repeated applications of Lemma 2, to have minimum probability over all achievable distributions.

Hence a polynomial algorithm for computing a minimum entropy coloring of (G, U) could be used to check the 3-colorability of any planar graph F, a problem known to be NP-complete [Sto73]. □

4 A polynomial case

We identify a special case of problem MinEntCol for which there exists a polynomial algorithm. Similar special cases were given for other coloring problems in [MP03, DdWMP02].

Theorem 2. *If $\alpha(G) \leq 2$, then there exists a polynomial algorithm for* MinEntCol*.*

Proof. A color class in any coloring of G is composed of one or two vertices. Each color class of size two identifies an edge in $E(\overline{G})$. The set of edges corresponding to the color classes of size two forms a matching in $\overline{G}$.

If a color class is composed of a single vertex v, then the contribution of this color to the overall entropy is $f(P(v))$. Otherwise, the contribution of a color is $f(P(v)+P(w))$, where v and w are the only two vertices in the color class. So if we denote by M the matching in $\overline{G}$ induced by a coloring of G, the entropy of this coloring with respect to P is

$$\begin{aligned}
& \sum_{v \notin V(M)} f(P(v)) + \sum_{\{v,w\} \in E(M)} f(P(v)+P(w)) \\
= & \sum_{v \in V(G)} f(P(v)) \\
& + \left(\sum_{\{v,w\} \in E(M)} f(P(v)+P(w)) - f(P(v)) - f(P(w)) \right) \\
= & \; H(X) + \sum_{e \in E(M)} \rho(e), \qquad (5)
\end{aligned}$$

where $X \sim P(x)$ and $\rho(\{v,w\}) = f(P(v)+P(w)) - f(P(v)) - f(P(w))$. Hence finding a minimum entropy coloring amounts to finding a minimum weight matching in $\overline{G}$, each edge e of which has weight $\rho(e)$. This can be done in $O(|V(G)|^3)$ time. □

Note that this class of graphs include complements of bipartite graphs, hence also complements of trees and paths, as well as complements of cycles.

5 Number of colors

In this section, we are interested in the number of colors required to minimize the entropy of a coloring. We give a variant of Brooks' theorem for the chromatic entropy and prove that there exist graph with small chromatic number and whose minimum entropy colorings require arbitrarily many colors.

Theorem 3 (Brooks 1941). *If G is a connected graph such that $G \neq K_n$ and $G \neq C_{2k+1}$, then $\chi(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of a vertex in G.*

Theorem 4. $\chi_H(G,P) \leq \Delta(G)+1$.

Proof. Let us assume that a minimum entropy coloring ϕ of (G,P) uses k colors, with $k > \Delta(G)+1$. Let us select a vertex x such that no other color has probability strictly less than $P(\phi^{-1}(\phi(x)))$. This vertex has at most $\Delta(G)$ neighbors, hence is adjacent to vertices with at most $\Delta(G)$ different colors. There is therefore at least one color j in which x can be recolored and for which $P(\phi^{-1}(j)) \geq P(\phi^{-1}(\phi(x)))$. From Lemma 2, recoloring x with color j can only decrease the entropy of the coloring, so ϕ has not minimum entropy and we have a contradiction. □

Theorem 5. *If G is a connected graph such that $G \neq K_n$ and $G \neq C_{2k+1}$, then $\chi_H(G, U) \leq \Delta(G)$, where U is the uniform distribution over $V(G)$.*

Proof. The proof closely follows the one given in Diestel [Die97] for Brooks' theorem.

Let $\Delta = \Delta(G)$. If $\Delta \leq 2$, either G is an odd cycle and we have nothing to show, or G is a path or an even cycle and the proposition is trivial. Hence we assume $\Delta \geq 3$. Let us consider a minimal entropy coloring ϕ of (G, U) with colors in the set $\{1, 2, \ldots, \Delta+1\}$. We show that if the $\Delta+1$ colors are used, then ϕ cannot have minimum entropy with respect to U, unless G is the complete graph.

Without loss of generality, we consider that color $\Delta+1$ has minimum weight: for all $1 \leq c \leq \Delta$, we have $P(c) \geq P(\Delta+1)$. Let us choose a vertex $v \in V(G)$ such that $\phi(v) = \Delta + 1$. v must be adjacent to Δ vertices colored with colors 1 to Δ, otherwise it could be recolored and, from Lemma 2, ϕ would not have minimum entropy. We denote by v_i the vertex adjacent to v and such that $\phi(v_i) = i$. Let F be defined as the graph induced by the vertices colored with colors $\{1, 2, \ldots, \Delta\}$, $F_{i,j}$ as the graph induced on F by the vertices colored with colors i or j, and C_{ij} (respectively C_{ji}) as the component of $F_{i,j}$ containing v_i (respectively v_j).

We first show the following:

$$C_{ij} \text{ is a path.} \tag{6}$$

First, v_i must have a single neighbor in C_{ij}. Otherwise, it could be recolored, with a color c different of i and j and v could in turn be recolored with color i. The probability mass $P(\Delta+1)$ would decrease by $1/n$, and $P(c)$ would increase by $1/n$. From Lemma 2 and the fact that $P(\Delta+1) \leq P(c)$, the entropy would decrease, and ϕ would not be optimal.

Let us assume that C_{ij} is not a path. Then there must be an inner vertex of C_{ij} having three identically colored neighbors. Let us define u as the first such vertex on the path from v_i in C_{ij}. u can be recolored with a color c different from i or j, since its neighbors cannot have more than $\Delta-2$ distinct colors. We can then perform the following steps:

1. recolor u with color c,
2. interchange colors i and j on the path $v_i P \mathring{u}$,
3. recolor v with color i.

We show that these steps can only decrease the entropy, and therefore that ϕ cannot be optimal unless C_{ij} is a path. We have to consider two cases: either $\phi(u) = i$ or $\phi(u) = j$.

If $\phi(u) = i$, then the path $v_i P \mathring{u}$ is even. Interchanging the colors i and j on this path does not change the probability masses $P(i)$ and $P(j)$. The probability $P(i)$ is decreased by $1/n$ when u is recolored, but increased by $1/n$ when v is recolored. Hence the overall sequence of changes leaves the probability $P(i)$ unchanged, while $P(\Delta + 1)$ is decreased by $1/n$, and $P(c)$ is increased by $1/n$. Since $P(\Delta + 1) \leq P(c)$, we have the conditions of Lemma 2 and the entropy can only decrease.

If $\phi(u) = j$, then the path $v_i P \mathring{u}$ is odd. Interchanging colors i and j on this path decreases $P(i)$ by $1/n$ and increases $P(j)$ by $1/n$. Recoloring u decreases $P(j)$ by $1/n$ and recoloring v increases $P(i)$ by $1/n$. So the probability masses $P(i)$ and $P(j)$ are left unchanged by the operations. Overall, we only have that $P(\Delta + 1)$ is decreased by $1/n$, and $P(c)$ is increased by $1/n$. Again, Lemma 2 holds and the entropy decreases.

$$C_{ij} = C_{ji} \text{ is a } v_i - v_j \text{ path.} \tag{7}$$

To show this, we assume that C_{ij} and C_{ji} are disjoint components of $F_{i,j}$. Then we can interchange colors i and j in C_{ij} and recolor v with color i. We again have two cases: either C_{ij} is an even path, and $P(\Delta + 1)$ is decreased by $1/n$ and $P(i)$ is increased by $1/n$, or C_{ij} is an odd path, and $P(\Delta + 1)$ is decreased by $1/n$ and $P(j)$ is increased by $1/n$. In both cases, Lemma 2 holds and the entropy decreases.

$$\text{For distinct } i, j, k \text{ we have } C_{ij} \cap C_{jk} = \{v_j\}. \tag{8}$$

Otherwise there would be a vertex $v_j \neq u \in C_{ij} \cap C_{jk}$ with $\phi(u) = j$ and two pairs of neighbors colored with j and k respectively. We could then apply the same three steps as in the proof of point (6).

Now if the neighbors v_i of v are pairwise adjacent, then G can only be the graph induced by v and $\{v_1, v_2, \ldots, v_\Delta\}$, because all vertices have maximum degree Δ. Hence G is the complete graph, and we do not have to show anything.

We may thus assume without loss of generality that $v_1 v_2 \notin E(G)$. Let u be the neighbor of v_1 in C_{12}, with $\phi(u) = 2$. Interchanging colors 1 and 3 in C_{13} we obtain a new coloring ϕ' of F. This coloring has the same entropy as ϕ, since C_{13} is an even path. We define v_i' and C_{ij}' with respect to the new coloring ϕ'. As a neighbor of $v_1 = v_3'$, u now lies in C_{23}', for $\phi(u) = \phi'(u) = 2$. By (8), however, the path $\mathring{v}_1 C_{12}$ retained its original coloring, so $u \in \mathring{v}_1 C_{12} \subset C_{12}'$. Hence $u \in C_{23}' \cap C_{12}'$, contradicting (8).

□

We now give a method for incrementing the value of χ_H in a probabilistic graph. Let (G, P) be any probabilistic graph. For each positive real number α, we define a new probabilistic graph $(\tilde{G}, \tilde{P})$ as follows. We construct $\tilde{G}$ from G by attaching a new edge to each vertex. That is to say, we let $V(\tilde{G}) = \{v, v' :$

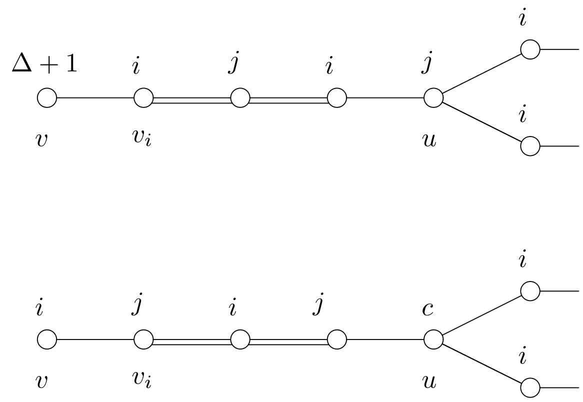


Figure 5: Illustration of the proof of proposition 6 (top: before performing the three recoloring steps, bottom: after).

$v \in V(G)\}$ and $E(\tilde{G}) = E(G) \cup \{\{v, v'\} : v \in V(G)\}$. Then we define $\tilde{P}$ by $\tilde{P}(v) = P(v)/\alpha$ and $\tilde{P}(v') = (1 - 1/\alpha)/n$ for each $v \in V(G)$, where $n = |V(G)|$.

Lemma 3. *Let* (G, P) *and* $(\tilde{G}, \tilde{P})$ *be as above. Then we have* $\chi_H(\tilde{G}, \tilde{P}) = \chi_H(G, P) + 1$*, provided that* α *is sufficiently large.*

Proof. Assume $\alpha > n + 1$. We can show that the unique maximum weight independent set in $\tilde{G}$ is $S^* = \{v' : v \in V(G)\}$. Indeed, suppose that S is a maximum weight independent set in $\tilde{G}$ containing some vertex v of the original graph G. Then $S' = S - v + v'$ is an independent set and we have

$$\tilde{P}(S') = \tilde{P}(S) - \frac{P(v)}{\alpha} + \frac{1 - \frac{1}{\alpha}}{n} > \tilde{P}(S),$$

a contradiction. This shows that every maximum weight independent set in $\tilde{G}$ is contained in S^*.

Consider the coloring ϕ_0 of $\tilde{G}$ that colors all vertices of S^* in one color and uses one color for each other vertex of $\tilde{G}$. The entropy of ϕ_0 is

$$H_0 = -\left(1 - \frac{1}{\alpha}\right) \log\left(1 - \frac{1}{\alpha}\right) - \sum_{v \in V(G)} \frac{P(v)}{\alpha} \log \frac{P(v)}{\alpha}$$

Now let ϕ denote any minimum entropy coloring of $\tilde{G}$. We claim that S^* has to be one of the color classes of ϕ. Otherwise, the maximum weight of a color class trivially satisfies

$$P_{\max} \leq 1 - \frac{1 - \frac{1}{\alpha}}{n} = \frac{n - 1 + \frac{1}{\alpha}}{n},$$

and we can bound the entropy of ϕ as follows:

$$H \geq -\log \frac{n-1+\frac{1}{\alpha}}{n} = \log \frac{n}{n-1+\frac{1}{\alpha}}.$$

The difference between this lower bound on H and H_0 tends to $\log n/(n-1)$ as α tends to ∞, contradicting the minimality of ϕ. Hence ϕ must color all vertices of S^* in the same color, provided α is large enough.

The entropy of ϕ can be written

$$H = \frac{1}{\alpha} H(\psi(X)) + H(\mathcal{B}(1/\alpha))$$

where ψ is the restriction of ϕ to G, $X \sim P(x)$ and $\mathcal{B}(1/\alpha)$ is a Bernoulli random variable with parameter $1/\alpha$. Hence, for each fixed, large enough α, the minimum entropy colorings of $(\tilde{G}, \tilde{P})$ are exactly those obtained from minimum entropy colorings of (G, P) by coloring all extra vertices in one extra color. In particular, we have $\chi_H(\tilde{G}, \tilde{P}) = \chi_H(G, P) + 1$ for α sufficiently large.

□

In the uniform case, we start with (G, U) and take α integer. The above transformation is made to obtain $(\tilde{G}, \tilde{U})$, where $\tilde{U}$ is not uniform, and $\chi_H(\tilde{G}, \tilde{U}) = \chi_H(G, U)+1$. Then, from Lemma 1, we can tranform $(\tilde{G}, \tilde{U})$ into a pair $(\tilde{G}', \tilde{U}')$ such that $\chi_H(\tilde{G}, \tilde{P}) = \chi_H(\tilde{G}', \tilde{U}')$ where $\tilde{U}'$ is the uniform distribution on the vertex set of $\tilde{G}'$. It is not difficult to verify that $\tilde{G}'$ can be directly obtained from G by attaching an $(\alpha - 1)$-star to each vertex.

This operation clearly preserves bipartiteness, acyclicity, and planarity. Therefore, by repeatedly applying this operation to a bipartite graph, a tree, or a planar graph we can increase χ_H by an arbitrary amount while keeping the chromatic number bounded by a small constant. Our final result follows.

Theorem 6. *$\chi_H(G, P)$ is not bounded by any function of $\chi(G)$, even if P is uniform and G is a bipartite graph, a tree, or a planar graph.*

6 Conclusion

A number of results can be generalized to other objective function computed from a feasible coloring ϕ. In fact, Theorem 2 applies to a large number of such functions, while Theorems 4, 5 and 6 rely only on Lemma 2. Hence these results apply to all objective functions of the form:

$$C(\phi) = \sum_{i \in \mathbb{N}} f \left(\sum_{v:\phi(v)=i} P(v) \right),$$

where $f(0) = 0$ and f is strictly concave. The minimum cardinality coloring problem can be casted in a similar way with

$$f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ otherwise.} \end{cases}$$

A number of issues remain open, in particular solving MinEntCol for various kinds of graphs, or finding approximation algorithms.

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