# On Minimum Entropy Graph Colorings

*IEEE International Symposium on Information Theory 2004*

**Jean Cardinal — Samuel Fiorini — Gilles Van Assche**

{jcardin,sfiorini,gvanassc}@ulb.ac.be.

Université Libre de Bruxelles (ULB)

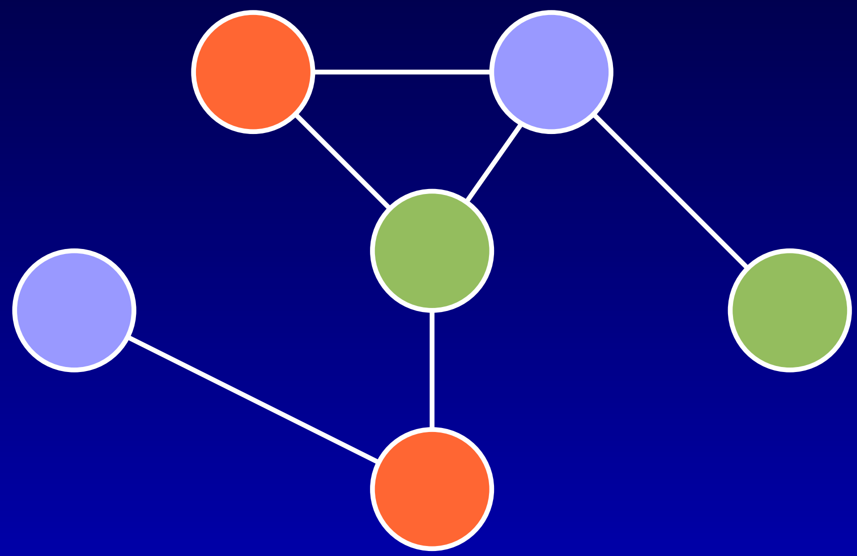
Brussels, Belgium

# Outline

- **Introduction**
  - Definitions
- Applications
- Complexity
- Number of Colors
- Conclusions

# Graph Coloring

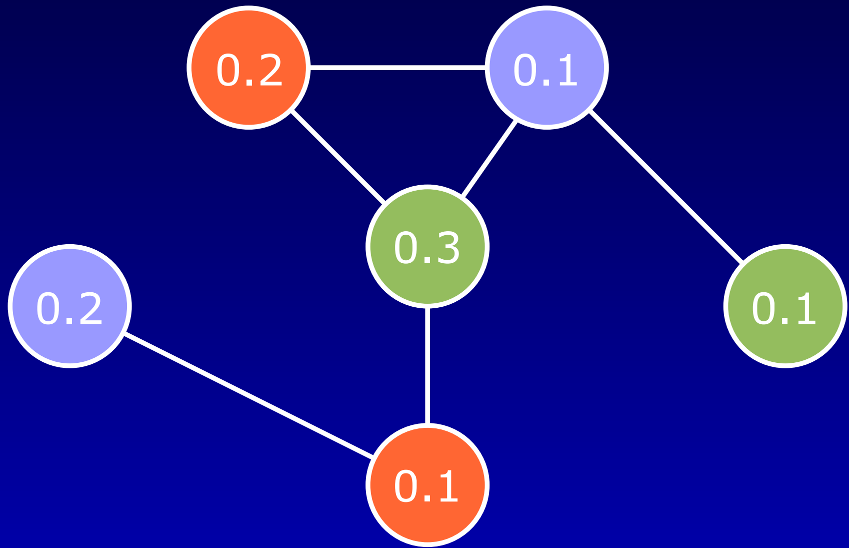
- **Coloring** $\varphi$ of $V$: $\{u, v\} \in E$ implies $\varphi(u) \neq \varphi(v)$



- **Chromatic number** $\chi(G) = \min_{\varphi} |\operatorname{Range}(\varphi)|$
- Many results about $\chi$
  - E.g., $G$ is planar $\Rightarrow \chi(G) \leq 4$

# Probabilistic Graphs

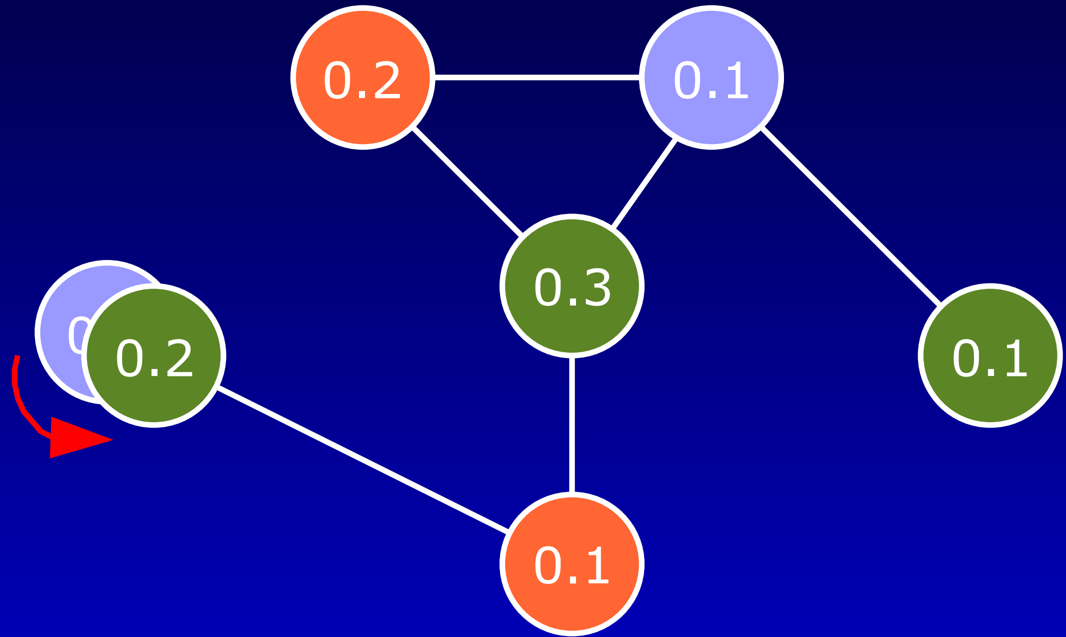
- **Probabilistic graph** $(G(V,E),P)$: probability distribution on vertices $V$: $P = \{p_i(v), v \in V\}$




- **Entropy of coloring**: $H(\varphi(X))$ if $X$ is a random variable on $V$ that follows $P$
  - Example: $H(\varphi(X)) = H(\{0.4, 0.3, 0.3\})$

# Chromatic Entropy

- **Chromatic entropy**: minimum entropy of any coloring, $H_\chi(G, P) = \min_\varphi H(\varphi(X))$




- Example: $H(\{0.6, 0.3, 0.1\}) < H(\{0.4, 0.3, 0.3\})$

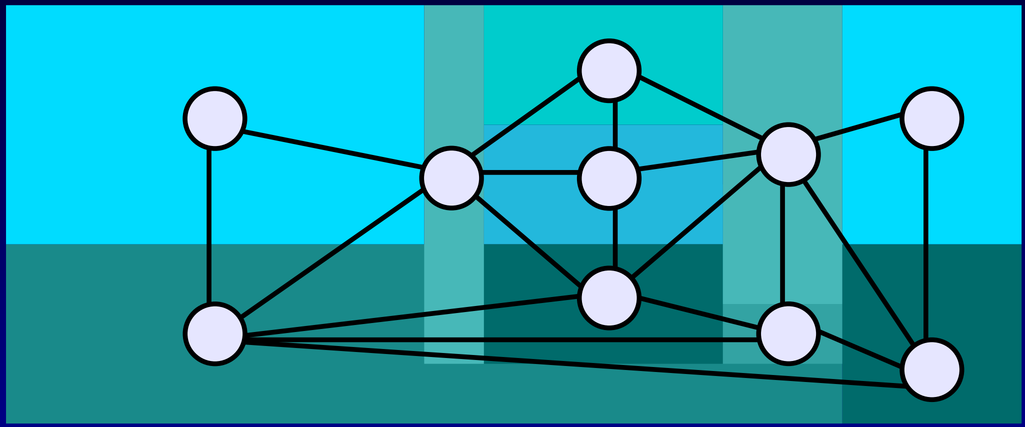
[1] Alon & Orlitsky, IEEE TIT 42(5), 1996

# Outline

- Introduction
- **Applications**
  - Compression of digital image partitions
  - Source coding with side information
- Complexity
- Number of Colors
- Conclusions

# Comp. of Image Partitions

- **Raster** image, segmented into **regions**



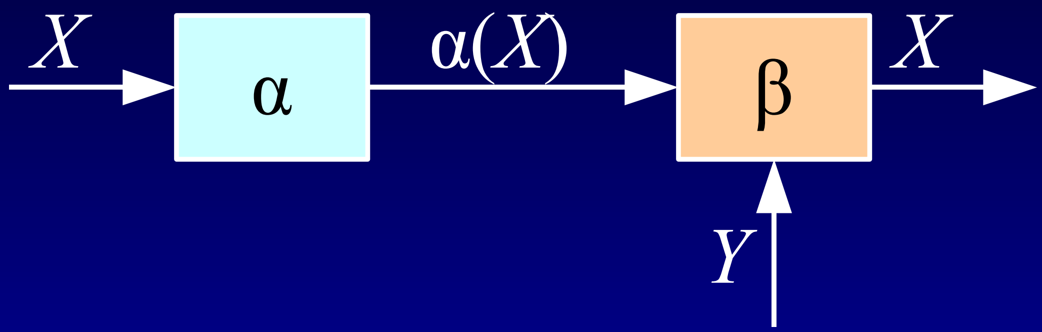
- Encoding of partition only: **compression**
  - Adjacency graph planar: up to 2 bits/pixel…
  - …but $H_\chi(G, P) < 2$ may be needed actually!
  - Sometimes 5 colors work better than 4


[2] Accame, De Natale & Granelli, Signal Proc., 80(6), 2000

[3] Agarwal & Belongie, Proc. IEEE ICIP, 2002

# Coding with Side Inform. (1/4)

- Source coding with side information known at the receiver




- No error is tolerated: **zero-error** coding required

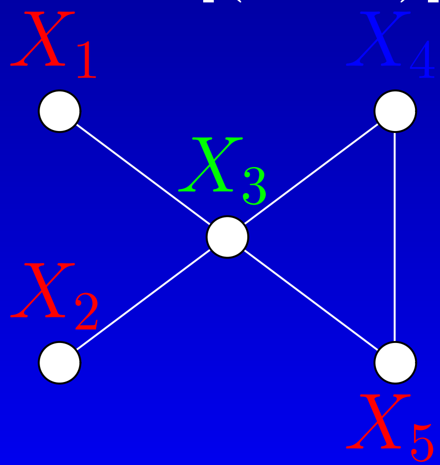
[4] Körner & Orlitsky, IEEE TIT 44(6), 1998

# Coding with Side Inform. (2/4)

- Example: encoding $X$ with $Y$ as side information

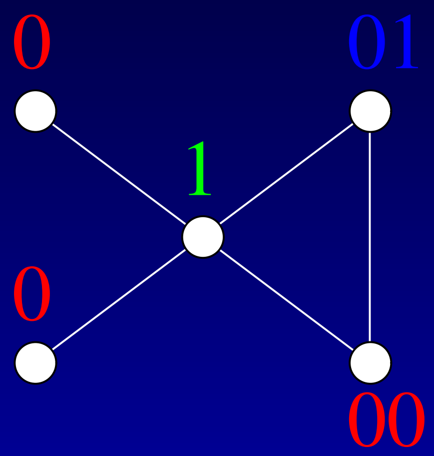
| $Y \backslash X$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|---|---|---|---|---|---|
| $Y_1$ | $1/7$ | | $1/7$ | | |
| $Y_2$ | | $1/7$ | $1/7$ | | |
| $Y_3$ | | | $1/7$ | $1/7$ | $1/7$ |

- **Characteristic graph** $G$: $V(G) = \mathcal{X}$, $x_1 x_2 \in E(G)$ iff $\exists y$: $\Pr[(x_1, y)] \Pr[(x_2, y)] > 0$



# Coding with Side Inform. (3/4)

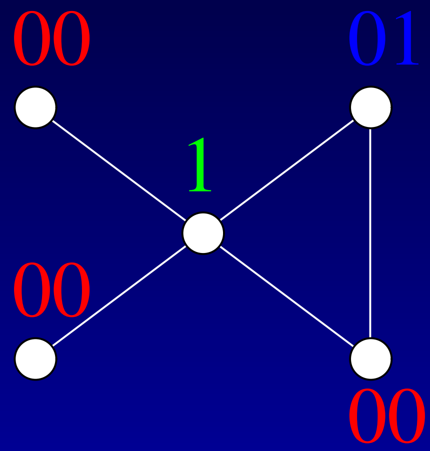
- **Restricted inputs**:
  $x_1 x_2 \in E(G) \Rightarrow \alpha(x_1)$ not a prefix of $\alpha(x_2)$




- Not prefix-free!
  - Prefix-free and unambiguous given any $Y = y$
- $L_{\mathrm{RI}} \leq H_\chi(G, X) + 1$
- $L_{\mathrm{RI},\infty} = \lim_{n\to\infty} \frac{1}{n} H_\chi(G^{\wedge n}, X^{(n)})$

# Coding with Side Inform. (4/4)

- **Unrestricted inputs**: globally prefix-free and $x_1 x_2 \in E(G) \Rightarrow \alpha(x_1) \neq \alpha(x_2)$




- Prefix-free without knowledge of $Y$ (more robust)
  - Unambiguous given any $Y = y$
- $H_\chi(G, X) \leq L_{\mathrm{UI}} \leq H_\chi(G, X) + 1$
- $L_{\mathrm{UI},\infty} = \lim_{n \to \infty} \frac{1}{n} H_\chi(G^{\vee n}, X^{(n)})$

# Outline

- Introduction
- Applications
- **Complexity**
  - On Maximum Weight Independent Sets
  - On Disjoint Components
  - Hardness of MINENTCOL
- Number of Colors
- Conclusions

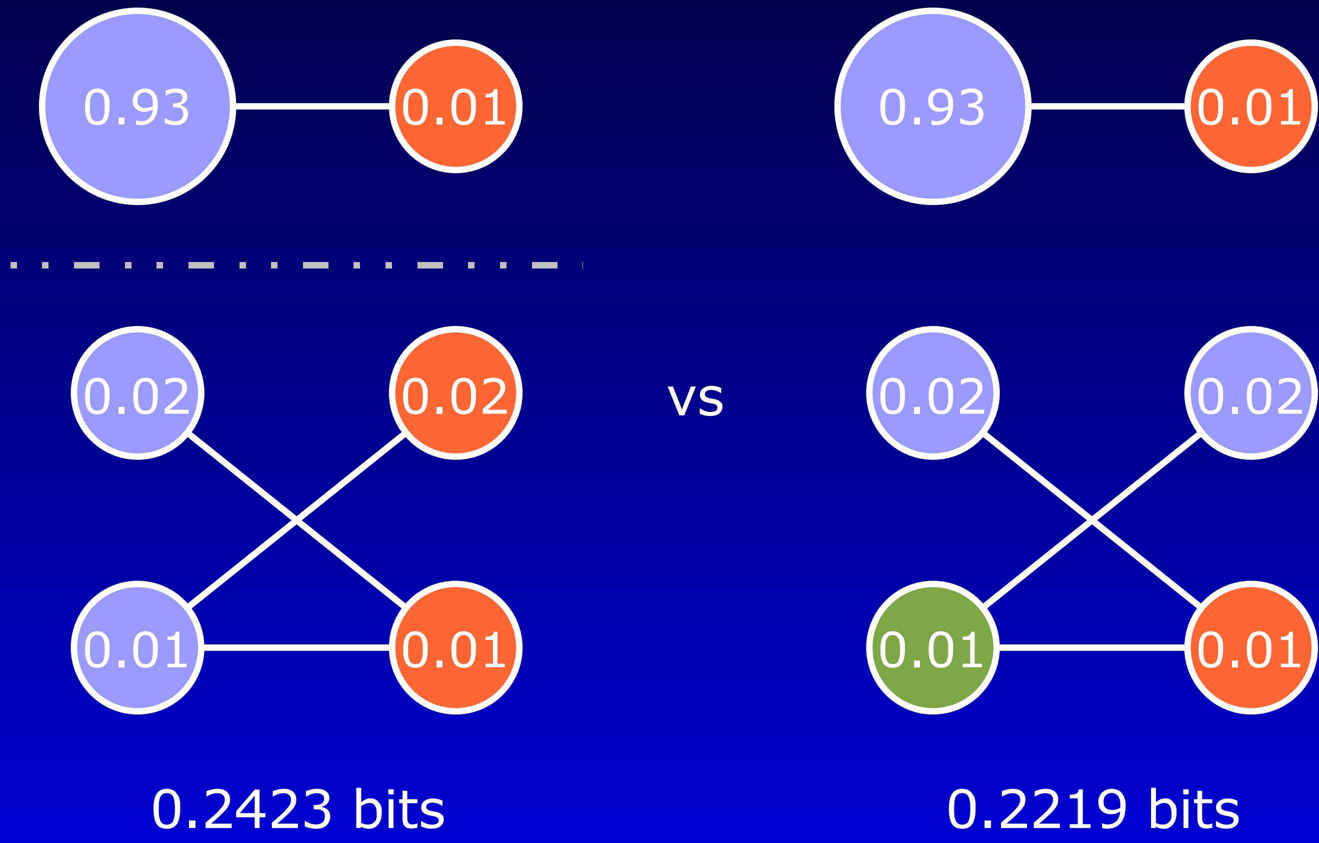
# On Max. Weight Indep. Sets

- Favor **large color classes**?


- The minimum entropy coloring does **not** always contain a maximum weight independent set!

# On Disjoint Components

- Can we optimize disjoint components **separately**?




- **No!**

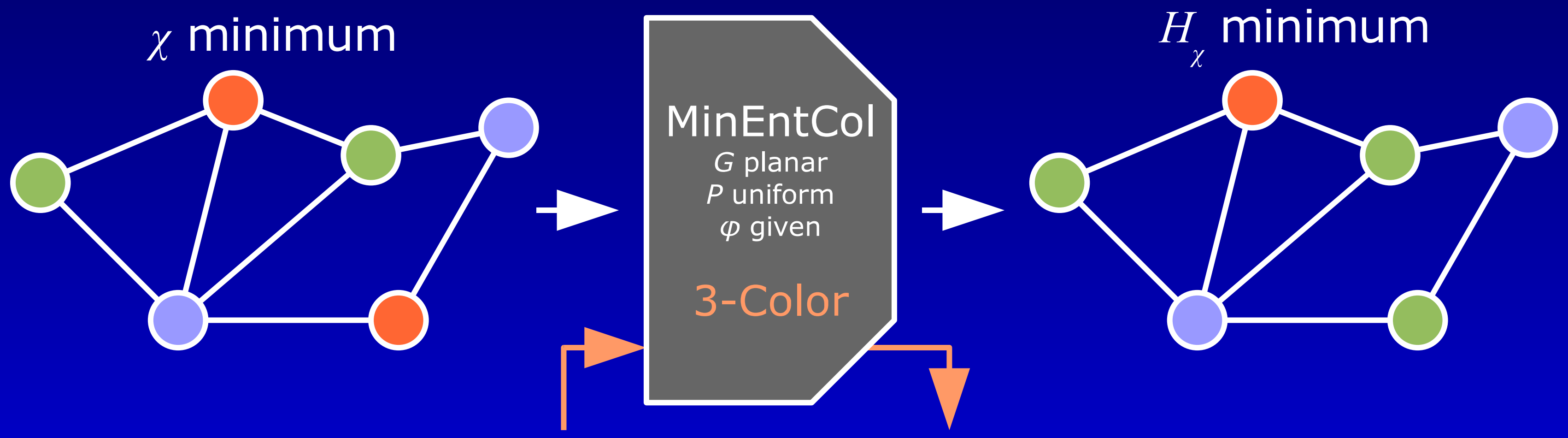
# Hardness of MinEntCol (1/2)

- MinEntCol:
  - Instance defined by $(G, P)$
  - Output: coloring $\varphi(V)$ such that $H(\varphi(X)) = H_\chi(G, P)$
- MinEntCol is **NP-hard**

[5] Zhao & Effros, Proc. IEEE DCC, 2003

# Hardness of MinEntCol (2/2)

- MinEntCol still NP-hard if restricted:
  - $G(V, E)$ is **planar**,
  - $P$ is the **uniform** distribution, and
  - $\varphi(V)$ that achieves $\chi(G)$ is **given** as input




- Proof by reduction to 3-colorability
- Finding $\chi$ and $H_\chi$ are different matters!

# Outline

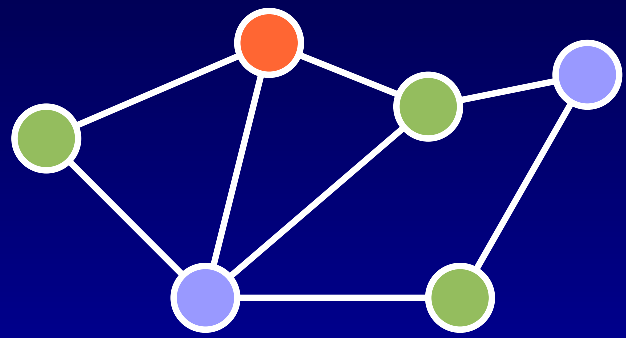
- Introduction
- Applications
- Complexity
- **Number of Colors**
  - Definition of $\chi_H$
  - Construction to Increase $\chi_H$
- Conclusions

# Number of Colors

- **Definition**: $\chi_H(G, P)$ is the minimum number of colors to achieve $H_\chi(G, P)$
- Simple bound: $\chi_H(G, P) \leq \Delta(G) + 1$, where $\Delta(G)$ is the max. degree of any vertex of $G$
- Questions:
  - Can $\chi_H(G, P) > \chi(G)$? **Yes!**
  - Does $\exists f : \chi_H(G, P) \leq f(\chi(G))$? **No!**

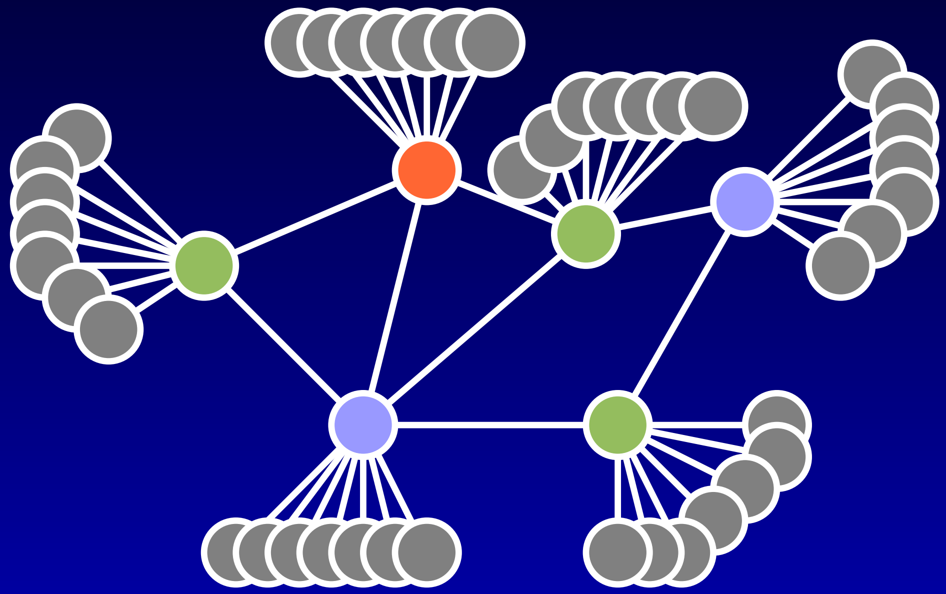
# Construction to Increase $\chi_H$

- Attach $n$ vertices to each vertex of $G$:



# Construction to Increase $\chi_H$

- Attach $n$ vertices to each vertex of $G$:



- For $n$ sufficiently large: new vertices need **one new color**
- Closed for bipartite graphs, trees, planar graphs
- Repeat it many times: $\chi_H(G, P) \gg \chi(G)$

# Outline

- Introduction
- Applications
- Complexity
- Number of Colors
- **Conclusions**

# Conclusions

- Entropy graph coloring: interesting problem with many applications
- **Results:**
  - MinEntCol is NP-hard, even if $G$ planar, $P$ uniform and min. coloring given
  - $\chi_H(G, P) \leq \Delta(G) + 1$
  - $\chi_H(G, P) \not\leq f(\chi(G))$
- **Recent results:**
  - Polynomial algorithm for graphs $G$ such that $\bar{G}$ is triangle-free
  - $G$ not complete nor odd cycle, $P$ uniform $\Rightarrow \chi_H(G, P) \leq \Delta(G)$ (variant of Brooks' th.)

# Conclusions

- **Open problems:**
  - Polynomial algorithm for other families of graphs? Cycles, bipartite graphs, trees?
  - Lower bounds on $\chi_H(G, P)$?
  - Source coding with side information: what about small error tolerance?

See `http://www.ulb.ac.be/di/publications/`