On Minimum Entropy Graph Colorings

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Outline

- **Introduction**
  - Definitions
- Applications
- Complexity
- Number of Colors
- Conclusions
Graph Coloring

• **Coloring** \( \varphi \) of \( V: \{u, v\} \in E \) implies \( \varphi(u) \neq \varphi(v) \)

\[ \begin{align*}
\text{Chromatic number} & \quad \chi(G) = \min_{\varphi} |\text{Range}(\varphi)| \\
\text{Many results about} \; \chi & \quad \text{E.g., } G \text{ is planar} \Rightarrow \chi(G) \leq 4
\end{align*} \]
Probabilistic Graphs

- **Probabilistic graph** \((G(V, E), P)\): probability distribution on vertices \(V\): \(P = \{p_i(v), v \in V\}\)

- **Entropy of coloring**: \(H(\varphi(X))\) if \(X\) is a random variable on \(V\) that follows \(P\)
  - Example: \(H(\varphi(X)) = H(\{0.4, 0.3, 0.3\})\)
**Chromatic Entropy**

- **Chromatic entropy**: minimum entropy of any coloring, \( H_\chi(G, P) = \min_\varphi H(\varphi(X)) \)

![Graph Image]

- Example: \( H(\{0.6, 0.3, 0.1\}) < H(\{0.4, 0.3, 0.3\}) \)

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Outline

• Introduction

• **Applications**
  • Compression of digital image partitions
  • Source coding with side information

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Comp. of Image Partitions

- **Raster** image, segmented into **regions**

- Encoding of partition only: **compression**
  - Adjacency graph planar: up to 2 bits/pixel...
  - …but \( H_\chi(G, P) < 2 \) may be needed actually!
  - Sometimes 5 colors work better than 4


Coding with Side Inform. (1/4)

- Source coding with side information known at the receiver

\[ X \xrightarrow{\alpha} \alpha(X) \xrightarrow{\beta} X \]

- No error is tolerated: **zero-error** coding required

Coding with Side Inform. (2/4)

- Example: encoding $X$ with $Y$ as side information

<table>
<thead>
<tr>
<th>$Y \setminus X$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>1/7</td>
<td>1/7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Y_2$</td>
<td></td>
<td>1/7</td>
<td>1/7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Y_3$</td>
<td></td>
<td></td>
<td>1/7</td>
<td>1/7</td>
<td>1/7</td>
</tr>
</tbody>
</table>

- Characteristic graph $G$: $V(G) = \mathcal{X}$,
  $x_1x_2 \in E(G)$ iff $\exists y$: $\Pr[(x_1, y)] \Pr[(x_2, y)] > 0$
Coding with Side Inform. (3/4)

- **Restricted inputs:**
  \[ x_1x_2 \in E(G) \Rightarrow \alpha(x_1) \text{ not a prefix of } \alpha(x_2) \]

- **Not prefix-free!**
  - Prefix-free and unambiguous given any \( Y = y \)
  - \( L_{RI} \leq H_{\chi}(G, X) + 1 \)
  - \( L_{RI,\infty} = \lim_{n \to \infty} \frac{1}{n} H_{\chi}(G^{\wedge n}, X^{(n)}) \)
Coding with Side Inform. (4/4)

- **Unrestricted inputs:** globally prefix-free and $x_1 x_2 \in E(G) \Rightarrow \alpha(x_1) \neq \alpha(x_2)$

\[
\begin{array}{c}
00 & 01 \\
\circ & \circ \\
1 & 01 \\
00 & 00 \\
\end{array}
\]

- Prefix-free without knowledge of $Y$ (more robust)
  - Unambiguous given any $Y = y$
  - $H_X(G, X) \leq L_{UI} \leq H_X(G, X) + 1$
  - $L_{UI, \infty} = \lim_{n \to \infty} \frac{1}{n} H_X(G^{\wedge n}, X^{(n)})$
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  - On Maximum Weight Independent Sets
  - On Disjoint Components
  - Hardness of MINENTCOL
- Number of Colors
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On Max. Weight Indep. Sets

- Favor large color classes?

![Graphs with different color classes and entropy values]

- The minimum entropy coloring does not always contain a maximum weight independent set!
On Disjoint Components

- Can we optimize disjoint components separately?

- No!
Hardness of MINENTCOL (1/2)

- **MINENTCOL**:  
  - Instance defined by \((G, P)\)
  - Output: coloring \(\varphi(V)\) such that \(H(\varphi(X)) = H_\chi(G, P)\)
- **MINENTCOL** is **NP-hard**

Hardness of MINENTCOL (2/2)

- MINENTCOL still NP-hard if restricted:
  - $G(V, E)$ is planar,
  - $P$ is the uniform distribution, and
  - $\varphi(V)$ that achieves $\chi(G)$ is given as input

- Proof by reduction to 3-colorability
- Finding $\chi$ and $H_\chi$ are different matters!
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• **Number of Colors**
  • Definition of $\chi_H$
  • Construction to Increase $\chi_H$
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Number of Colors

- **Definition**: $\chi_H(G, P)$ is the minimum number of colors to achieve $H_\chi(G, P)$

- Simple bound: $\chi_H(G, P) \leq \Delta(G) + 1$, where $\Delta(G)$ is the max. degree of any vertex of $G$

- Questions:
  - Can $\chi_H(G, P) > \chi(G)$? **Yes!**
  - Does $\exists f : \chi_H(G, P) \leq f(\chi(G))$? **No!**
Construction to Increase $\chi_H$

- Attach $n$ vertices to each vertex of $G$:
Construction to Increase $\chi_H$

- Attach $n$ vertices to each vertex of $G$:

- For $n$ sufficiently large: new vertices need one new color
- Closed for bipartite graphs, trees, planar graphs
- Repeat it many times: $\chi_H(G, P) \gg \chi(G)$
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Conclusions

- Entropy graph coloring: interesting problem with many applications

**Results:**
- MINENTCOL is NP-hard, even if $G$ planar, $P$ uniform and min. coloring given
  - $\chi_H(G, P) \leq \Delta(G) + 1$
  - $\chi_H(G, P) \not\leq f(\chi(G))$

**Recent results:**
- Polynomial algorithm for graphs $G$ such that $\bar{G}$ is triangle-free
- $G$ not complete nor odd cycle, $P$ uniform
  $\implies \chi_H(G, P) \leq \Delta(G)$ (variant of Brooks’ th.)
Conclusions

- **Open problems:**
  - Polynomial algorithm for other families of graphs? Cycles, bipartite graphs, trees?
  - Lower bounds on $\chi_H(G, P)$?
  - Source coding with side information: what about small error tolerance?

See http://www.ulb.ac.be/di/publications/