On Minimum Entropy Graph Colorings

IEEE International Symposium on Information Theory 2004

Jean Cardinal — Samuel Fiorini — Gilles Van Assche

{jcardin,sfiorini,gvanassc}@ulb.ac.be.

Université Libre de Bruxelles (ULB)

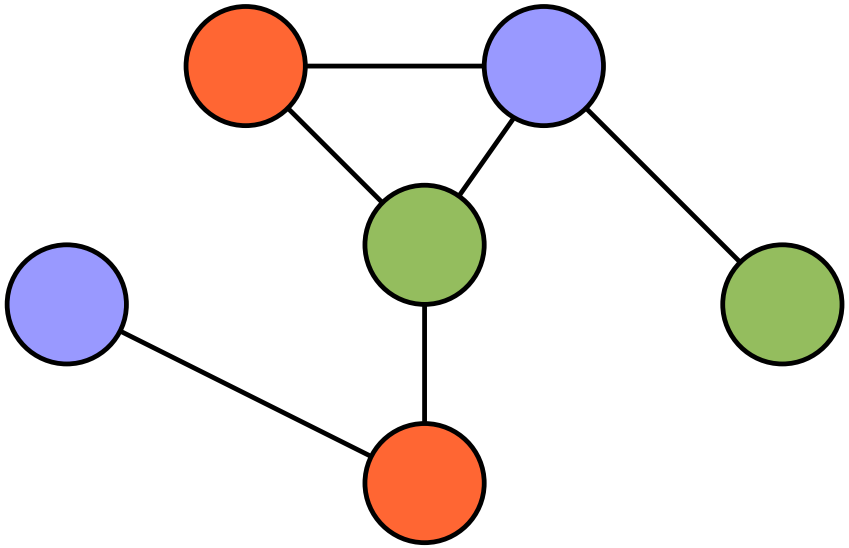
Brussels, Belgium

Outline

- **Introduction**
  - Definitions
- Applications
- Complexity
- Number of Colors
- Conclusions

Graph Coloring

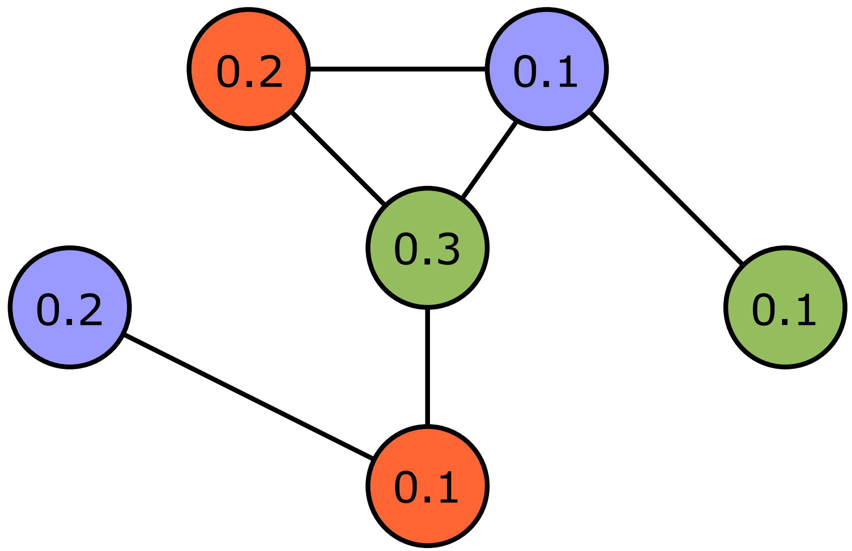
- **Coloring** φ of V: $\{u, v\} \in E$ implies $\varphi(u) \neq \varphi(v)$



- **Chromatic number** $\chi(G) = \min_{\varphi} |\,\mathrm{Range}(\varphi)|$
- Many results about χ
  - E.g., G is planar $\Rightarrow \chi(G) \leq 4$

Probabilistic Graphs

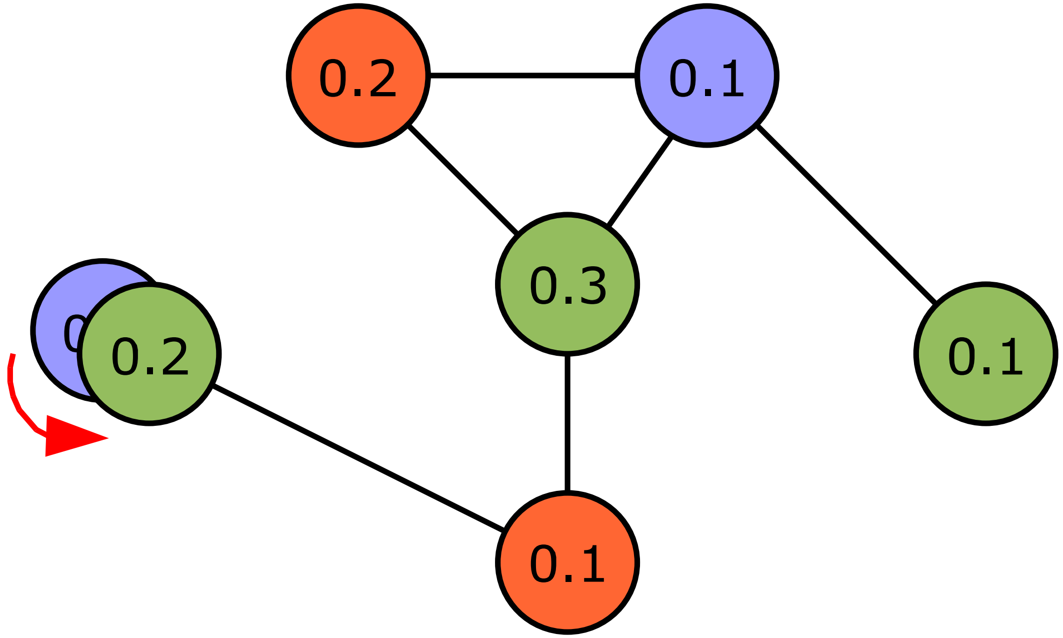
- **Probabilistic graph** $(G(V,E),P)$: probability distribution on vertices V: $P = \{p_i(v), v \in V\}$



- **Entropy of coloring**: $H(\varphi(X))$ if X is a random variable on V that follows P
  - Example: $H(\varphi(X)) = H(\{0.4, 0.3, 0.3\})$

Chromatic Entropy

- **Chromatic entropy**: minimum entropy of any coloring, $H_\chi(G, P) = \min_\varphi H(\varphi(X))$



- Example: $H(\{0.6, 0.3, 0.1\}) < H(\{0.4, 0.3, 0.3\})$

[1] Alon & Orlitsky, IEEE TIT 42(5), 1996

Outline

- Introduction
- **Applications**
  - Compression of digital image partitions
  - Source coding with side information
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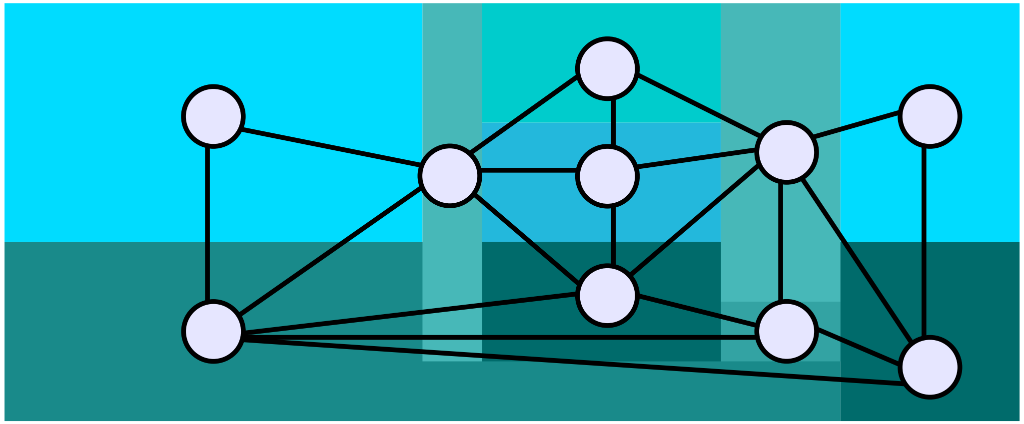
Comp. of Image Partitions

- **Raster** image, segmented into **regions**



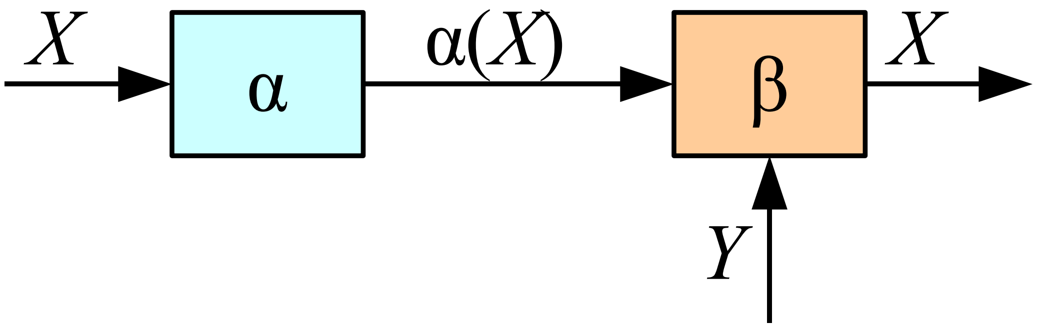
- Encoding of partition only: **compression**
  - Adjacency graph planar: up to 2 bits/pixel…
  - …but $H_\chi(G, P) < 2$ may be needed actually!
  - Sometimes 5 colors work better than 4

[2] Accame, De Natale & Granelli, Signal Proc., 80(6), 2000

[3] Agarwal & Belongie, Proc. IEEE ICIP, 2002

Coding with Side Inform. (1/4)

- Source coding with side information known at the receiver



- No error is tolerated: **zero-error** coding required

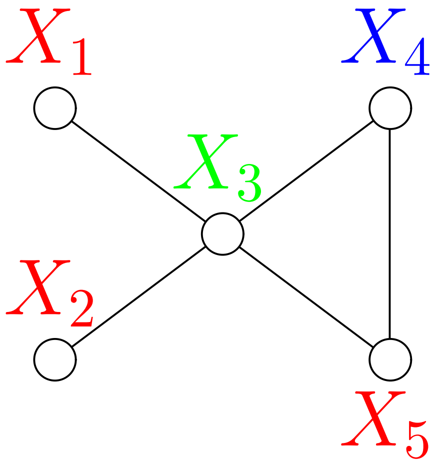
[4] Körner & Orlitsky, IEEE TIT 44(6), 1998

Coding with Side Inform. (2/4)

- Example: encoding X with Y as side information

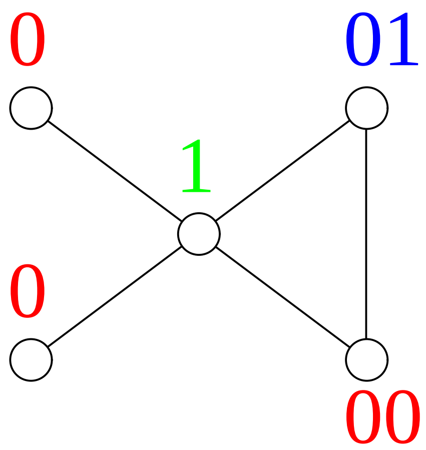
$Y \backslash X$	X_1	X_2	X_3	X_4	X_5
Y_1	$1/7$		$1/7$		
Y_2		$1/7$	$1/7$		
Y_3			$1/7$	$1/7$	$1/7$

- **Characteristic graph** G: $V(G) = \mathcal{X}$, $x_1 x_2 \in E(G)$ iff $\exists y$: $\Pr[(x_1, y)] \Pr[(x_2, y)] > 0$



Coding with Side Inform. (3/4)

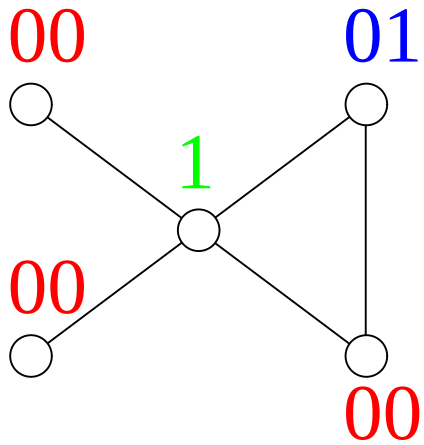
- **Restricted inputs**:
  $x_1 x_2 \in E(G) \Rightarrow \alpha(x_1)$ not a prefix of $\alpha(x_2)$



- Not prefix-free!
  - Prefix-free and unambiguous given any $Y = y$
- $L_{\text{RI}} \leq H_\chi(G, X) + 1$
- $L_{\text{RI},\infty} = \lim_{n\to\infty} \frac{1}{n} H_\chi(G^{\wedge n}, X^{(n)})$

Coding with Side Inform. (4/4)

- **Unrestricted inputs**: globally prefix-free and $x_1 x_2 \in E(G) \Rightarrow \alpha(x_1) \neq \alpha(x_2)$



- Prefix-free without knowledge of Y (more robust)
  - Unambiguous given any $Y = y$
- $H_\chi(G, X) \leq L_{\text{UI}} \leq H_\chi(G, X) + 1$
- $L_{\text{UI},\infty} = \lim_{n\to\infty} \frac{1}{n} H_\chi(G^{\vee n}, X^{(n)})$

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- **Complexity**
  - On Maximum Weight Independent Sets
  - On Disjoint Components
  - Hardness of MINENTCOL
- Number of Colors
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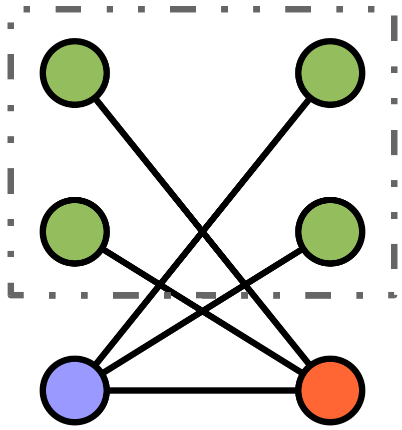
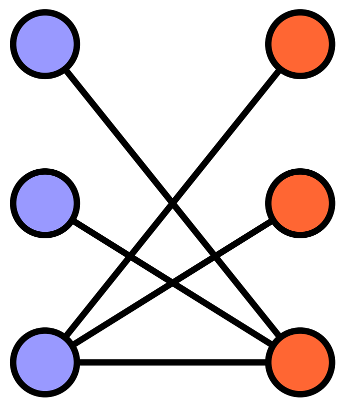
On Max. Weight Indep. Sets

- Favor **large color classes**?



vs



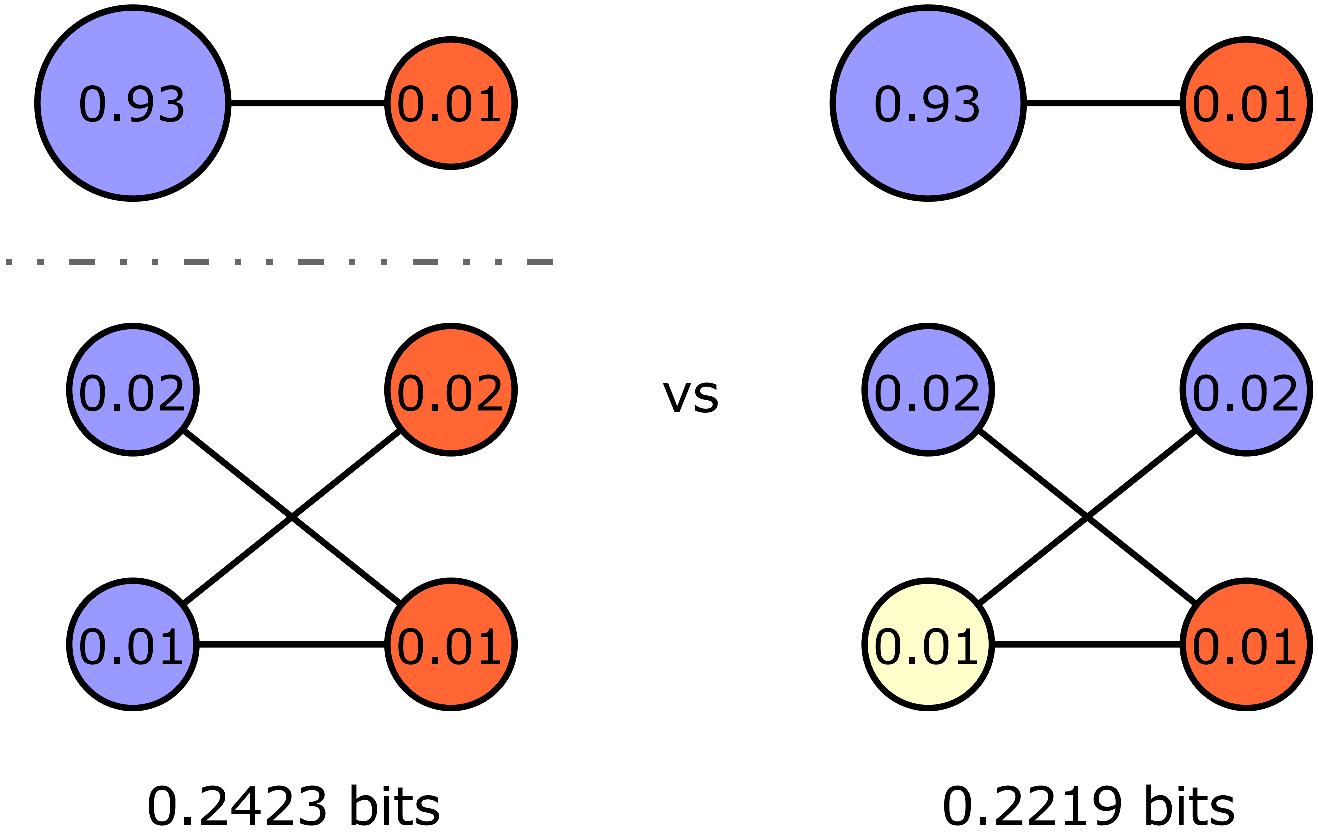
1.2516 bits

1 bit

- The minimum entropy coloring does **not** always contain a maximum weight independent set!

On Disjoint Components

- Can we optimize disjoint components **separately**?



- **No!**

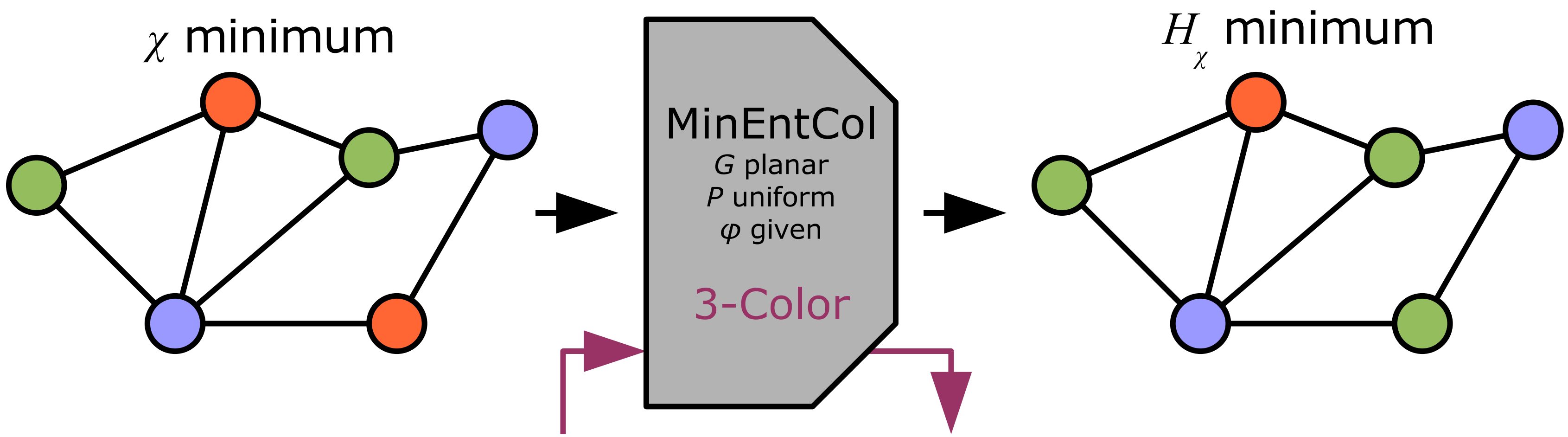
Hardness of MINENTCOL (1/2)

- MINENTCOL:
  - Instance defined by (G, P)
  - Output: coloring $\varphi(V)$ such that $H(\varphi(X)) = H_\chi(G, P)$
- MINENTCOL is **NP-hard**

[5] Zhao & Effros, Proc. IEEE DCC, 2003

Hardness of MINENTCOL (2/2)

- MINENTCOL still NP-hard if restricted:
  - $G(V, E)$ is **planar**,
  - P is the **uniform** distribution, and
  - $\varphi(V)$ that achieves $\chi(G)$ is **given** as input



- Proof by reduction to 3-colorability
- Finding χ and H_χ are different matters!

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  - Definition of χ_H
  - Construction to Increase χ_H
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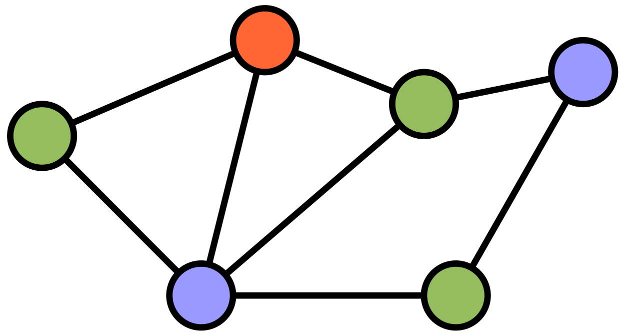
Number of Colors

- **Definition**: $\chi_H(G, P)$ is the minimum number of colors to achieve $H_\chi(G, P)$
- Simple bound: $\chi_H(G, P) \le \Delta(G) + 1$, where $\Delta(G)$ is the max. degree of any vertex of G
- Questions:
  - Can $\chi_H(G, P) > \chi(G)$? **Yes!**
  - Does $\exists f : \chi_H(G, P) \le f(\chi(G))$? **No!**

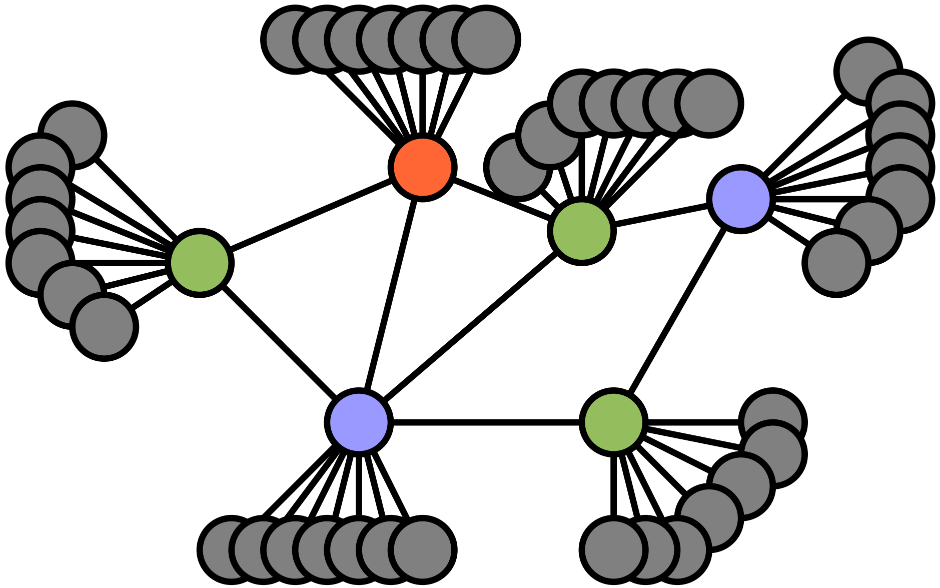
Construction to Increase χ_H

- Attach n vertices to each vertex of G:



Construction to Increase χ_H

- Attach n vertices to each vertex of G:



- For n sufficiently large: new vertices need **one new color**
- Closed for bipartite graphs, trees, planar graphs
- Repeat it many times: $\chi_H(G, P) \gg \chi(G)$

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Conclusions

- Entropy graph coloring: interesting problem with many applications
- **Results:**
  - MinEntCol is NP-hard, even if G planar, P uniform and min. coloring given
  - $\chi_H(G, P) \leq \Delta(G) + 1$
  - $\chi_H(G, P) \not\leq f(\chi(G))$
- **Recent results:**
  - Polynomial algorithm for graphs G such that $\bar{G}$ is triangle-free
  - G not complete nor odd cycle, P uniform $\Rightarrow \chi_H(G, P) \leq \Delta(G)$ (variant of Brooks' th.)

Conclusions

- **Open problems:**
  - Polynomial algorithm for other families of graphs? Cycles, bipartite graphs, trees?
  - Lower bounds on $\chi_H(G, P)$?
  - Source coding with side information: what about small error tolerance?

See `http://www.ulb.ac.be/di/publications/`